Robust Invertibility of Switched Linear Systems

Aneel Tanwani and Daniel Liberzon

***Abstract*— In this paper, we address the effects of uncertainties in output measurements and initial conditions on invertibility of the switched systems – the problem concerned with the recovery of the input and the switching signal using the output and the initial state. By computing the reachable sets and maximal error in the propagation of state trajectories through the inverse system, we derive conditions under which it is possible to recover the exact switching signal over a certain time interval, provided the uncertainties are bounded in some sense. In addition, we discuss separately the case where each subsystem is minimum phase and it is possible to recover the exact switching signal globally in time. The input, though, is recoverable only up to a neighborhood of the original input.**

I. Introduction

In the past decade, much literature has been published that relates to the structural properties of the switched systems; most recent of which is the problem of invertibility [1], [2]. Viewing the switching signal as an additional exogenous signal, invertibility of switched systems deals with the reconstruction of the input and the switching signal using the measured output and the initial state.

One of the main drawbacks of the algorithms used in the reconstruction of the input and the switching signal using the invertibility approach is that they require precise knowledge of the output and the initial state. Due to physical limitations of the sensors and non-uniform/unpredictable operating conditions of the system, it is often the case that these two quantities (the output and the initial condition) are not precisely known. Thus, it is natural to ask whether it is possible to recover the unknown switching signal and the input with disturbances in the measurement of the output and lack of precise knowledge of the initial state. Motivated by this practical setup, this paper deals with the reconstruction of the input and the switching signal when there are uncertainties in the output measurements and the initial condition.

We restrict our attention to linear switched systems described as:

$$\Gamma_\sigma : \begin{cases} \dot{x} = A_\sigma x + B_\sigma u, \\ y = C_\sigma x + D_\sigma u. \end{cases} \tag{1}$$

The switching signal σ takes values in a finite index set $\mathcal{P}$. For each $p \in \mathcal{P}$, $A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times m}$, $C_p \in \mathbb{R}^{m\times n}$, $D_p \in \mathbb{R}^{m\times m}$so that $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^m$; also, state variable $x \in \mathbb{R}^n$. The input and output dimensions are assumed to be same so that the system is square. Further, $u \in \mathcal{F}^n$, so that the output $y \in \mathcal{F}^n$, where $\mathcal{F}^n$ denotes the subset of piecewise right-continuous functions whose elements are n-times continuously differentiable between any two consecutive discontinuities. We denote by $\Gamma^O_{p,x_0}(u)$ the output of subsystem p with initial condition x_0 and input u. The Euclidean norm of a vector is denoted by $|\cdot|$ and the induced norm of a matrix by $\|\cdot\|$. For two matrices, A_1 and A_2, $\mathrm{col}(A_1, A_2) := [A_1^\top, A_2^\top]^\top$. For a vector z in Euclidean space, and a positive scalar r, let $\mathcal{B}_r(z) := \{y : |y - z| \le r\}$.

Problem Setup: With $y : \mathbb{R} \to \mathbb{R}^m$ as the exact output of the system, let $Y^k := \mathrm{col}(y, \dot{y}, \ddot{y}, \cdots, y^{(k)})$, $k \in \mathbb{N}$, denote the vector comprising the exact output and its first k-derivatives. For brevity, $Y := Y^n$. Both y and Y^k are considered to be unknown. Let $\widehat{Y}^k := \mathrm{col}(\hat{y}, \dot{\hat{y}}, \ddot{\hat{y}}, \cdots, \hat{y}^{(k)})$ denote the imprecise estimate of the output and its derivatives obtained by inaccurate measurements and numerical differentiation. Several useful techniques for obtaining the estimates of the derivatives, even for noisy signals, have been discussed in the literature, see [3] and references therein. It is assumed that for each t, the uncertainty in the measurement of the output and its derivatives is bounded by some fixed and known number $\varrho > 0$, that is, $|Y(t) - \widehat{Y}(t)| \le \varrho$. Also, the exact knowledge of the initial state $x_0 := x(t_0)$ is no longer assumed, instead x_0 is assumed to be contained in a known compact and convex set $\mathcal{R}_{t_0}$, so that $\hat{x}_0 \in \mathcal{R}_{t_0}$ is an initial estimate of x_0. *Our objective is to: (a) find conditions on subsystem dynamics and a deterministic function $\widetilde{\Sigma}^{-1}(\hat{x}_0, \hat{y})$ that reconstructs the original value of σ over some time interval, (b) compute the maximum error between the actual and the reconstructed input, (c) find conditions under which $\widetilde{\Sigma}^{-1}(\hat{x}_0, \hat{y})$ yields the actual value of σ at all times for a particular class of systems.*

II. Background and Preliminaries

Invertibility of switched system (1) requires each subsystem to be invertible and the identification of the active mode [1]. To check the former property, i.e., invertibility of a subsystem, one uses Silverman's structure algorithm [4], [5]; this paper, however, uses the notations developed in a terse version of the structure algorithm given in [1]. If a subsystem is invertible, the structure algorithm leads to the construction of an inverse subsystem that reconstructs the original input using the state and the output values. For mode identification in (1), we first develop a relationship between the output and the state for each subsystem and then utilize it to determine the active subsystem at each time instant. This relationship is characterized by the range theorem [5] and the characterization uses certain operators L_p and W_p, which are obtained by applying the structure algorithm to each subsystem Γ_p. The formula for L_p in terms of system data appears in [1], whereas the expression for W_p is developed

Coordinated Science Laboratory, Department of Electrical and Computer Engineering, University of Illinois at Urbana Champaign, Urbana, IL 61801, USA. Email: {tanwani2,liberzon}@illinois.edu

This work was supported by NSF under grant ECCS-0821153.

in the Appendix. The exact expressions of these operators are not required in the understanding of this paper, and we refer the reader to [1] and the Appendix if such formulae are sought. The following example helps illustrate how these operators show up in computations:

Example 1: Consider a non-switched linear SISO system

$$\Gamma_1 : \{ \ \dot{x} = \left[\begin{smallmatrix} -1 & 1 \\ 1 & 1 \end{smallmatrix}\right] x + \left[\begin{smallmatrix} 0.5 \\ 0 \end{smallmatrix}\right] u, \text{and } y = [0 \ 0.5]x.$$

Clearly, y and $\dot{y}$ are independent of the input, that is, $\left(\begin{smallmatrix} y \\ \dot{y} \end{smallmatrix}\right) = \left[\begin{smallmatrix} 0 & 0.5 \\ 0.5 & 0.5 \end{smallmatrix}\right] x$, or equivalently $[I_{2\times 2} \ 0_{2\times 1}]Y = \left[\begin{smallmatrix} 0 & 0.5 \\ 0.5 & 0.5 \end{smallmatrix}\right] x$, where $Y := \mathrm{col}(y, \dot{y}, \ddot{y})$. So, we let $W_1 := [I_{2\times 2} \ 0_{2\times 1}]$, and $L_1 := \left[\begin{smallmatrix} 0 & 0.5 \\ 0.5 & 0.5 \end{smallmatrix}\right]$ and the relation $W_1 Y(t) = L_1 x(t)$ holds $\forall\, t \geq t_0$. Computing the expression for $\ddot{y}$, solving it for u, and plugging the resultant back into the original dynamics yields the corresponding inverse system,

$$\Gamma_1^{-1} : \{ \ \dot{\hat{x}} = \left[\begin{smallmatrix} -1 & -1 \\ 1 & 1 \end{smallmatrix}\right] \hat{x} + \left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right] \hat{\ddot{y}}; u = [0 \ -4]\hat{x} + 4\hat{\ddot{y}}. \qquad \triangleleft$$

For the sake of clear presentation, we introduce the following assumptions to state a simplified version of the range theorem in Proposition 1 which characterizes the relationship between the output and the state for a subsystem Γ_p.

Assumption 1: Throughout the paper, it is assumed that:

1) each subsystem Γ_p, $p \in \mathcal{P}$, is invertible;
2) and the inputs are such that the output produced by each subsystem is n-times differentiable (i.e., $\mathcal{C}^n$).

Proposition 1: Consider system (1) with initial state x_0. If Assumption 1 holds and $y \in \mathcal{C}^n([t_0, t_1), \mathbb{R}^m)$, then there exists an input u such that $y = \Gamma^O_{p,x_0}(u)$ if and only if $W_p Y \in \mathcal{C}^0([t_0, t_1), \mathbb{R}^m)$ and $W_p Y(t_0) = L_p x_0$. $\triangleleft$

It is also noted from the structure algorithm that regardless of what the input is, the output and the state are related by the equation $W_p Y(t) = L_p x(t)$, for all $t \geq t_0$ when Γ_p is active, and not just at the initial time t_0. When dealing with the switched systems, this is the fundamental idea employed in mode detection and it also leads to the following concept of switch-singular pairs [1].

Definition 2: Let $x_0 \in \mathbb{R}^n$ and $y \in \mathcal{C}^n$ be an $\mathbb{R}^m$-valued function on some time interval. The pair (x_0, y) is a *switch-singular pair* of the two subsystems Γ_p, Γ_q if there exist u_1, u_2 such that $\Gamma^O_{p,x_0}(u_{1[t_0,t_0+\epsilon)}) = \Gamma^O_{q,x_0}(u_{2[t_0,t_0+\epsilon)}) = y_{[t_0,t_0+\epsilon)}$, for some $\epsilon > 0$. $\triangleleft$

Essentially, if a state and an output (the time domain can be arbitrary) form a switch-singular pair, then there exist inputs for the two subsystems to produce that same output starting from that same initial state. Under Assumption 1, it follows from Definition 2 and Proposition 1 that (x_0, y) is a switch-singular pair for Γ_p, Γ_q if, and only if,

$$W_{pq} Y(t_0) = L_{pq} x_0, \tag{2}$$

where $W_{pq} := \mathrm{col}(W_p, W_q)$, $L_{pq} := \mathrm{col}(L_p, L_q)$, $x_0 = x(t_0)$, and t_0 is the initial time of y. This condition for verifying the existence of switch-singular pairs can be checked easily for a certain class of outputs using a rank condition. According to [1, Lemma 3], if y is such that $W_{pq} Y(t) \neq 0$, for any $t \geq t_0$, then there exist no switch-singular pairs (x_0, y) between subsystems Γ_p and Γ_q if, and only if,

$$\mathrm{rank}\, [L_{pq} \ W_{pq}] = \mathrm{rank}\, L_{pq} + \mathrm{rank}\, W_{pq}. \tag{3}$$

If $\mathcal{L}_{pq}$ and $\mathcal{W}_{pq}$ denote the range spaces of the matrices L_{pq} and W_{pq} respectively, then geometrically, condition (3) is equivalent to saying that $\mathcal{L}_{pq} \cap \mathcal{W}_{pq} = \{0\}$.

Next, let $\overline{\mathcal{Y}}$ be the set of piecewise smooth functions such that if $y \in \overline{\mathcal{Y}}$, then $W_{pq} Y(t_0) \neq 0$ for all $p \neq q$, $p, q \in \mathcal{P}$. It has been shown in [1] that, for the output set $\overline{\mathcal{Y}}$, a switched system is invertible if and only if all subsystems are invertible and subsystem dynamics are such that there exist no switch-singular pairs among them. So, if Assumption 1 and (3) hold then the switched system is invertible for the output set $\overline{\mathcal{Y}}$.

In case a switched system is invertible, a switched inverse system can be constructed to recover the input and the switching signal σ from the given output and the initial state. Towards that end, define the *index-inversion function* $\overline{\Sigma}^{-1} : \mathbb{R}^n \times \overline{\mathcal{Y}} \to \mathcal{P}$ as:

$$\overline{\Sigma}^{-1}(x_0, y) = \{p : W_p Y(t_0) = L_p x_0\}. \tag{4}$$

Having determined the mode using (4), the corresponding inverse system is activated to recover the input. Thus, an inverse switched system Γ_σ^{-1}, with initial condition x_0, is implemented as follows:

$$\sigma(t) = \overline{\Sigma}^{-1}(x(t), y_{[t,t+\epsilon)}), \tag{5a}$$
$$\dot{x} = \widehat{A}_{\sigma(t)} x + \widehat{B}_{\sigma(t)} Y, \tag{5b}$$
$$u = \widehat{C}_{\sigma(t)} x + \widehat{D}_{\sigma(t)} Y \tag{5c}$$

where $\widehat{A}_\sigma := (A - B\overline{D}_\alpha^{-1}\overline{C}_\alpha)_\sigma$, $\widehat{B}_\sigma := (B\overline{D}_\alpha^{-1} V)_\sigma$, $\widehat{C}_\sigma := (-\overline{D}_\alpha^{-1}\overline{C}_\alpha)_\sigma$, and $\widehat{D}_\sigma := (\overline{D}_\alpha^{-1} V)_\sigma$. The matrices $\overline{C}_\alpha, \overline{D}_\alpha, V$ are defined for each subsystem through the structure algorithm; formulae for $\overline{C}_\alpha, \overline{D}_\alpha$ are given in [1], and the expression for V is developed in the Appendix.

III. Inversion under Uncertainties

In the problem setup, it is assumed that x_0 and Y are unknown, and we work with their respective estimates $\hat{x}_0 \in \mathcal{R}_{t_0}$, and $\widehat{Y}$. So, instead of (5), the following equations are utilized, with the initial condition $\hat{x}_0$, to get an estimate of the actual state trajectory and the input appearing in (1), which are now denoted by $\hat{x}$ and $\hat{u}$ respectively.

$$\dot{\hat{x}} = \widehat{A}_\sigma \hat{x} + \widehat{B}_\sigma \widehat{Y}, \tag{6a}$$
$$\hat{u} = \widehat{C}_\sigma \hat{x} + \widehat{D}_\sigma \widehat{Y}. \tag{6b}$$

In (6), the switching signal σ remains unknown and the remainder of this section concentrates on recovering the switching signal using $\mathcal{R}_{t_0}$ and $\widehat{Y}$. In §III-A, we use the concept of reachable sets to compute σ; in §III-B, simpler computations are devised to reconstruct σ for the case $\mathcal{R}_{t_0} = \mathcal{B}_{\delta_0}(\hat{x}_0)$ and some known δ_0; and §III-C quantifies the error between the actual input u and its estimate $\hat{u}$.

A. Switching Signal Recovery using Reachable Sets

In the presence of uncertainties in the output and the state values, the most natural extension of the mode identification method described in the previous section is to compute the set that contains $x(t)$ and look at the intersection of the image of this set under the map L_p with $W_p(\mathcal{B}_\varrho(\widehat{Y}(t)))$.

The subsystem Γ_p is declared active if the corresponding intersection is non-empty. In the sequel, we formalize this argument to show how the switching signal can be recovered using that approach.

1) Reachable Sets: It follows from Fillipov's theorem for linear time varying systems [6] that if the initial condition lies in a compact and convex set, then the reachable set $\mathcal{R}_{t_1} := \{x(t_1) : x(t) \text{ solves (6a) with } t \in [t_0, t_1], x(t_0) \in \mathcal{R}_{t_0}\}$ is compact, convex and varies continuously with t_1. Several methods for computing the set $\mathcal{R}_t$ are available in literature [7] and we assume that $\mathcal{R}_t$ can be computed. Note that the operator L_p obtained from the structure algorithm is linear and continuous. Since $\mathcal{R}_t$ is a compact and convex set at each time t, the set $L_p(\mathcal{R}_t)$ is also compact and convex.

2) Index Matching Function $\widehat{\Sigma}^{-1}$: To compute the value of the switching signal $\sigma(t)$ using the index-inversion function (4), we find p for which $W_p Y(t) = L_p x(t)$, or alternatively $|W_p Y(t) - L_p x(t)| = 0$. Since $Y(t)$ and $x(t)$ are no longer available, this condition cannot be verified anymore. A new function, that recovers the value of the switching signal is computed using the following proposition.

Proposition 3: For system (1), if there exists an input u such that $\Gamma^O_{p,x_0}(u) = y$ over an interval $[t_0, t_1)$, and $|Y(t) - \widehat{Y}(t)| \leq \varrho$, then $|L_p x(t) - W_p \widehat{Y}(t)| \leq \|W_p\| \varrho$, for each $t \in [t_0, t_1)$.

Proof: Since $\Gamma^O_{p,x_0}(u) = y$ over the interval $[t_0, t_1)$ for some input u, it follows from Proposition 1 that $L_p x(t) = W_p Y(t)$ for each $t \in [t_0, t_1)$. So that, $|L_p x(t) - W_p \widehat{Y}(t)| = |\cancel{L_p x(t)} - \cancel{W_p Y(t)} + W_p Y(t) - W_p \widehat{Y}(t)| = |W_p(Y(t) - \widehat{Y}(t))| \leq \|W_p\| \, |Y(t) - \widehat{Y}(t)| \leq \|W_p\| \varrho$. ■

Note that, rather than the exact value of $x(t)$, it is only known that $x(t) \in \mathcal{R}_t$. If $\mathcal{Z}_p(t) := L_p(\mathcal{R}_t)$, we define the distance between the set $\mathcal{Z}_p(t)$ and the vector $W_p \widehat{Y}(t)$ to be:

$$d_p(t) := \min_{z \in \mathcal{Z}_p(t)} |W_p \widehat{Y}(t) - z| \cdot \tag{7}$$

The set $\mathcal{Z}_p(t)$ is compact, convex and contains $L_p x(t)$, so there exists a unique solution to this optimization problem according to the *projection theorem* [8]. Use of Proposition 3 guarantees that if Γ_p produces the output at time t, then the distance between the set $\mathcal{Z}_p(t)$ and $W_p \widehat{Y}(t)$ is less than $\|W_p\| \varrho$. This motivates us to introduce the following definition of *index-matching function*:

$$\widehat{\Sigma}^{-1}(\mathcal{R}_t, \hat{y}_{[t,t+\epsilon)}) := \{p \,|\, d_p(t) \leq \|W_p\| \varrho\}. \tag{8}$$

Next, in Proposition 4, it is shown that the distance function (7) is continuous locally in time and that the value of the index-matching function (8) coincides with the original switching signal. The proof uses continuity of reachable sets and has been omitted due to space constraints.

Proposition 4: Consider the switched system (1) with initial condition contained in a compact, convex set $\mathcal{R}_{t_0}$, and measured output $\hat{y}$ over some time interval. Assume that:

1) there is a unique $p \in \mathcal{P}$ such that $d_p(t_0) \leq \|W_p\| \varrho$,
2) for all $q \neq p$, $d_q(t_0) > 3\|W_q\| \varrho$,

then there exists $\rho > 0$ such that $\sigma(t) = \widehat{\Sigma}^{-1}(\mathcal{R}_t, \hat{y}_{[t,t+\epsilon)})$ for all $t \in [t_0, t_0 + \rho)$, where $\widehat{\Sigma}^{-1}$ is defined in (8). ◁

Proposition 4 also suggests that there exists a lower bound on the time interval for which the switching signal can be recovered. If the conditions of this proposition are also satisfied at time $t_0 + \rho$, then there exists $\overline{\rho} > 0$ such that the switching signal can be recovered over the interval $[t_0, t_0 + \rho + \overline{\rho})$. Thus, larger intervals can be obtained by applying Proposition 4 inductively. The switching signal $\sigma(\cdot)$ is recovered by letting $\sigma(t) = \widehat{\Sigma}^{-1}(\mathcal{R}_t, \hat{y}_{[t,t+\epsilon)})$ and this definition leads to the recovery of switching signal over an interval $[t_0, T)$, where $T := \min\{t \geq t_0 \,|\, \exists p, q \in \mathcal{P} \text{ s.t. } d_p(t) \leq \|W_p\| \varrho \text{ and } d_q(t) \leq \|W_q\| \varrho\}$. We now illustrate this result with an example.

Example 2: Consider a switched system with two modes, i.e., $\mathcal{P} = \{1, 2\}$; where subsystem Γ_1 and its inverse are defined in Example 1, Γ_2 and its inverse are given below:

$$\Gamma_2 \begin{cases} \dot{x} = \begin{bmatrix} -1 & 1 \\ 1 & \frac{-1}{2} \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{cases} ; \; \Gamma_2^{-1} \begin{cases} \dot{\hat{x}} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{4} \\ 1 & \frac{-1}{2} \end{bmatrix} \hat{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \hat{\dot{y}} \\ \hat{u} = \begin{bmatrix} \frac{3}{4} & \frac{-5}{8} \end{bmatrix} \hat{x} + \frac{1}{2} \hat{\dot{y}} \end{cases}$$

For this example, $W_1 = W_2 = [I_{2\times 2} \; 0_{2\times 1}]$; $L_1 = \left[\begin{smallmatrix} 0 & 0.5 \\ 0.5 & 0.5 \end{smallmatrix}\right]$ and $L_2 := \left[\begin{smallmatrix} 0 & 1 \\ 1 & -0.5 \end{smallmatrix}\right]$. Γ_1, Γ_2 are invertible and the conditions in Proposition 4 hold for $\varrho = 0.25$ and $\mathcal{R}_0 = \{x : (x - 10)^\top (x - 10) = 1)\}$ for an arbitrarily chosen output y.

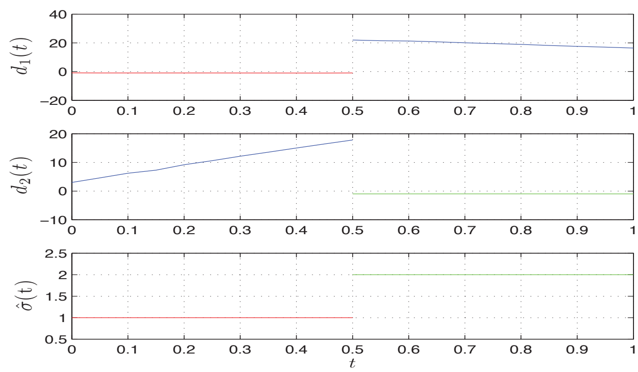

Fig. 1. The distance functions $d_1(t)$, $d_2(t)$ and the corresponding $\sigma(t)$.

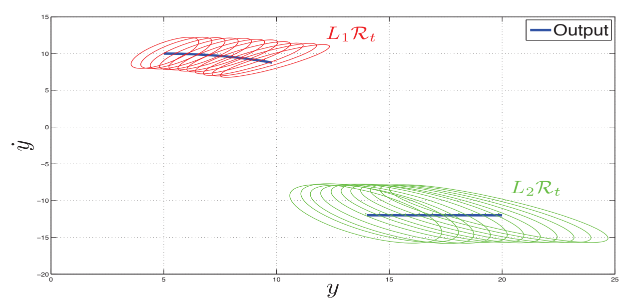

Fig. 2. The sets $\mathcal{Z}_1(t)$, $\mathcal{Z}_2(t)$, and the output trajectory in $(y, \dot{y})$ space.

Fig. 1 shows the corresponding distance functions; and the switching signal obtained by comparing these two distance functions. It can be seen that $\sigma(t) = p, p = 1, 2$, when $d_p(t)$ is near zero. Fig. 2 gives an insight into the values of these distance functions by plotting the sets $\mathcal{Z}_p(t) := L_p \mathcal{R}_t, p = 1, 2$, and the output trajectory in $(y, \dot{y})$-plane. ◁

B. Approximate Reachable Sets

In the previous section, we recovered the switching signal using the index-matching function $\widehat{\Sigma}^{-1}$, whose arguments were the measured output and the reachable set at each point in time, and furthermore the evaluation of this function involved the solution to an optimization problem at each instant in time. Clearly, this approach is computationally very expensive. In this section, we let $\mathcal{R}_{t_0} = \mathcal{B}_{\delta_0}(\hat{x}_0)$ and derive an alternative simpler formula for the recovery of switching signal with the help of certain approximations, which relieves the computational burden enormously. The drawback, however, is that the interval over which the switching signal is recovered is smaller. We start off with the definition of $(\mathcal{R}, \varrho)$ *switch-singular pair*:

Definition 5: Let $x_0 = x(t_0)$ be contained in a compact set $\mathcal{R} \subset \mathbb{R}^n$, and y be an $\mathbb{R}^m$-valued function over some time interval with $Y_0 := Y(t_0)$. We say that (x_0, y) forms an $(\mathcal{R}, \varrho)$ switch-singular pair for subsystems Γ_p, Γ_q if for the given $\varrho > 0$, there exist $x_1, x_2 \in \mathcal{R}$ and $Y_1, Y_2 \in \mathcal{B}_\varrho(Y_0)$ such that $L_p x_1 = W_p Y_1$ and $L_q x_2 = W_q Y_2$. $\lhd$

In the sequel, we will also refer to $(\mathcal{R}, \varrho)$ switch-singular pair as the *weak switch-singular pair* when $\mathcal{R}$ and ϱ need not be specified.

1) Gap between Subspaces: To study the existence of weak switch-singular pairs, we introduce the notion of minimal gap between the subspaces.

Definition 6: Let $\mathcal{M}, \mathcal{N}$ be two subspaces of an Euclidean space. The *minimal gap* $\alpha(\mathcal{M}, \mathcal{N})$ is defined as:

$$\alpha(\mathcal{M}, \mathcal{N}) = \alpha(\mathcal{N}, \mathcal{M}) := \min\{\widehat{\alpha}(\mathcal{M}, \mathcal{N}), \widehat{\alpha}(\mathcal{N}, \mathcal{M})\}$$

where $\widehat{\alpha}(\mathcal{M}, \mathcal{N}) := \min_{|x|=1, x\in\mathcal{M}} d(x, \mathcal{N})$. $\lhd$

The notion of minimum gap between subspaces has appeared in [9], [10] for spaces other than Euclidean spaces.

Proposition 7 (Computation of $\widehat{\alpha}(\mathcal{M}, \mathcal{N})$): Let $\Pi_\mathcal{N}$ denote the orthogonal projection on $\mathcal{N}$ and matrix M be such that its columns are orthonormal vectors that span $\mathcal{M}$, then

$$\widehat{\alpha}(\mathcal{M}, \mathcal{N})^2 = \min_{|x|=1, x\in\mathcal{M}} d^2(x, \mathcal{N}) = 1 - \|\Pi_\mathcal{N} M\|^2 \qquad (9)$$

Proof: Using the projection theorem [8], the square of the distance between a point x and a subspace $\mathcal{N}$ is given by $|x|^2 - |\Pi_\mathcal{N} x|^2$. The desired expression can now be derived:

$$\begin{aligned}
&\min_{|x|=1, x\in\mathcal{M}} d^2(x, \mathcal{N}) = \min_{|x|=1, x\in\mathcal{M}} \left\{|x|^2 - |\Pi_\mathcal{N} x|^2\right\} \\
&= 1 - \max_{|x|=1, x\in\mathcal{M}} |\Pi_\mathcal{N} x|^2 = 1 - \max_{|Mz|=1} |\Pi_\mathcal{N} M z|^2 \\
&= 1 - \max_{|z|=1} |\Pi_\mathcal{N} M z|^2 = 1 - \|\Pi_\mathcal{N} M\|^2. \qquad \blacksquare
\end{aligned}$$

Note that $\alpha(\mathcal{M}, \mathcal{N}) = 0$ if and only if $\mathcal{M} \cap \mathcal{N} \neq \{0\}$, and $\alpha(\mathcal{M}, \mathcal{N}) = 1$ if and only if $\mathcal{M}$ and $\mathcal{N}$ are mutually orthogonal to each other. Roughly speaking, $\alpha(\mathcal{M}, \mathcal{N})$ measures the sine of minimum angle between $\mathcal{M}$ and $\mathcal{N}$.

Corollary 8: Suppose $\mathcal{M}, \mathcal{N}$ are two subspaces such that $\mathcal{M} \cap \mathcal{N} = \{0\}$. Given $x \in \mathcal{M}$, $z \in \mathcal{N}$, $|x - z| < \epsilon$ only if $|x| < \frac{\epsilon}{\alpha(\mathcal{M}, \mathcal{N})}$.

Proof: For $x \neq 0$, $x \in \mathcal{M}$ can be written as $x = cy$ where $y \in \mathcal{M}$ has unit norm, and $c \in \mathbb{R}$. Note that $|x| = |c|$. Using the reverse triangle inequality, we obtain:

$$\begin{aligned}
\epsilon > |x - z| \geq ||x| - |z|| \geq |c| \left||y| - \left|\frac{z}{c}\right|\right| \geq |c| d(y, \mathcal{N}) \\
\geq |c| \inf_{|y|=1, y\in\mathcal{M}} d(y, \mathcal{N}) = |c|\widehat{\alpha}(\mathcal{M}, \mathcal{N}) \geq |c|\alpha(\mathcal{M}, \mathcal{N}),
\end{aligned}$$

whence the desired result follows. $\blacksquare$

2) Necessary Conditions for Weak Switch-Singular Pairs: If for a given $\hat{x}_0$ and $\hat{y}$, there exist $x_0 \in \mathcal{B}_{\delta_0}(\hat{x}_0)$ and $Y_0 \in \mathcal{B}_\varrho(\widehat{Y}_0)$, $\widehat{Y}_0 := \widehat{Y}(t_0)$, such that $L_p x_0 = W_p Y_0$, i.e., Γ_p produces the output y with initial condition x_0, then

$$\begin{aligned}
|L_p \hat{x}_0 - W_p \widehat{Y}_0| &\leq |L_p \hat{x}_0 - L_p x_0| + \cancel{|L_p x_0 - W_p Y_0|} \\
&+ |W_p Y_0 - W_p \widehat{Y}_0| \leq \|L_p\| \delta_0 + \|W_p\| \varrho.
\end{aligned}$$

In particular, if $(\hat{x}_0, \hat{y})$ forms an $(\mathcal{B}_{\delta_0}(\hat{x}_0), \varrho)$ switch-singular pair then,

$$\begin{aligned}
|L_{pq} \hat{x}_0 - W_{pq} \widehat{Y}_0| &\leq |L_p \hat{x}_0 - W_p \widehat{Y}_0| + |L_q \hat{x}_0 - W_q \widehat{Y}_0| \\
&\leq (\|L_p\| + \|L_q\|)\delta_0 + (\|W_p\| + \|W_q\|)\varrho \\
&=: \kappa^0_{pq}.
\end{aligned}$$

This leads to the following necessary conditions for weak switch-singular pairs.

Proposition 9: If $\mathcal{L}_{pq} \cap \mathcal{W}_{pq} = \{0\}$, then $(\hat{x}_0, \hat{y})$ forms a $(\mathcal{B}_{\delta_0}(\hat{x}_0), \varrho)$ switch-singular pair for subsystems Γ_p and Γ_q, $p, q \in \mathcal{P}$ only if $|L_{pq} \hat{x}_0| \leq \frac{\kappa^0_{pq}}{\alpha(\mathcal{L}_{pq}, \mathcal{W}_{pq})}$ and $|W_{pq} \widehat{Y}_0| \leq \frac{\kappa^0_{pq}}{\alpha(\mathcal{L}_{pq}, \mathcal{W}_{pq})}$.

Proof: This is a straightforward consequence of Corollary 8 applied with $\mathcal{M} := \mathcal{L}_{pq}$ and $\mathcal{N} := \mathcal{W}_{pq}$. $\blacksquare$

The above proposition, thus, gives two necessary conditions under which subsystems Γ_p, Γ_q, $p, q \in \mathcal{P}$, may form weak switch-singular pairs.

Example 3: Consider the second order SISO switched system given in Example 2. The columns of $W_{12} := \mathrm{col}(W_1, W_2)$ and $L_{12} := \mathrm{col}(L_1, L_2)$ span two dimensional subspaces of $\mathbb{R}^4$ and it can be verified that their intersection is the null vector. In terms of orthonormal basis, we can write $\mathcal{W}_{12} = \mathrm{span}\left\{\mathrm{col}(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0), \mathrm{col}(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})\right\}$; $\mathcal{L}_{12} = \mathrm{span}\left\{\mathrm{col}(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}, 0), \mathrm{col}(\frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{2}{\sqrt{7}}, \frac{-1}{\sqrt{7}})\right\}$. Since both W_{12} and L_{12} comprise linearly independent columns, they are left-invertible and the orthogonal projection can be written in terms of the left-pseudo inverse (denoted by †), that is, $\Pi_{\mathcal{L}_{12}} = L_{12} L_{12}^\dagger = L_{12}(L_{12}^\top L_{12})^{-1} L_{12}^\top$, and $\Pi_{\mathcal{W}_{12}} = W_{12} W_{12}^\dagger = W_{12}(W_{12}^\top W_{12})^{-1} W_{12}^\top$. From these matrices we can now compute the gap between $\mathcal{L}_{12}$ and $\mathcal{W}_{12}$ using (9):

$$\alpha(\mathcal{W}_{12}, \mathcal{L}_{12}) = \widehat{\alpha}(\mathcal{W}_{12}, \mathcal{L}_{12}) = \widehat{\alpha}(\mathcal{L}_{12}, \mathcal{W}_{12}) = 0.1368.$$

With $\varrho = \delta_0 = 0.25$, we get $\kappa^0_{12} = 1.0224$. Considering the data of Example 1 at initial time t_0 with $\hat{x}_0 = \mathrm{col}(10, 10)$, we have $|W_{pq} \widehat{Y}_0| = 11.45 > \frac{\kappa^0_{pq}}{\alpha(\mathcal{L}_{pq}, \mathcal{W}_{pq})} = 7.02$, and $|L_{pq} \hat{x}_0| = 11.18$. Both necessary conditions are violated, so there are no weak switch-singular pairs at time t_0. $\lhd$

3) Spherical Approximation and Elimination of Switch-Singular Pairs over an Interval: Proposition 9 provides necessary conditions for the existence of weak switch-singular pairs at time instant t_0 when the uncertainty in the initial state is given by a ball of radius δ_0. Our goal now is to determine whether, under certain conditions, it is possible to rule out the existence of weak switch-singular pairs over some time interval. Note that, even though the output is changing with time, there is a constant upper bound on the uncertainties in the output ϱ, whereas the uncertainty in the state variable, denoted by $\delta(t)$, is a function of time. The value of $\delta(t)$ is basically obtained from the norm of the error vector $\tilde{x} = x - \hat{x}$, the dynamics for which are obtained as a difference of (5b) and (6a),

$$\dot{\tilde{x}}(t) = \widehat{A}_\sigma \tilde{x}(t) + \widehat{B}_\sigma w(t), \tag{10}$$

where $w(t) := Y(t) - \widehat{Y}(t)$, and $|w(t)| \leq \varrho$ for each $t \geq t_0$. With $|\tilde{x}(t_0)| \leq \delta_0 := \delta(t_0)$, an upper bound on $\delta(t)$ is obtained by solving (10) analytically for $\tilde{x}$. To guarantee that the state $\hat{x}(t)$ does not form $(\mathcal{B}_{\delta(t)}(\hat{x}(t)), \varrho)$ switch-singular pairs with the output $\hat{y}(t)$, one must verify that, at time t, the following inequality holds:

$$|L_{pq}\hat{x}(t) - W_{pq}\widehat{Y}(t)| \geq \kappa_{pq}^t \tag{11}$$

where $\kappa_{pq}^t := (\|L_p\| + \|L_q\|)\delta(t) + (\|W_p\| + \|W_q\|)\varrho$. The lower bound on t, for which (11) holds, is given in Theorem 10 as it specifies the length of a time interval during which the output and the state do not form the weak switch-singular pairs. Avoiding the details due to space constraints, we introduce some notation to state the result only. For each $p \in \mathcal{P}$, there exists $\lambda_p \neq 0, a_p \in \mathbb{R}$ such that $\|\exp(\widehat{A}_p t)\| \leq e^{(a_p + \lambda_p t)}$. Define $\lambda := \max_{p\in\mathcal{P}} \lambda_p$, $a := \max_{p\in\mathcal{P}} a_p$, $b := \max_{p\in\mathcal{P}} \|\widehat{B}_p\|$, and

$$\Omega := \max_{p,q\in\mathcal{P}} \frac{\alpha(\mathcal{L}_{pq}, \mathcal{W}_{pq})\beta - (\|W_p\| + \|W_q\|)\varrho}{(\|L_p\| + \|L_q\|)}. \tag{12}$$

Theorem 10: Consider switched system (1), and let $|\widehat{Y}(t) - Y(t)| \leq \varrho$ for each $t \geq t_0$. Moreover, assume that

1) condition (3) holds (i.e., $\mathcal{L}_{pq} \cap \mathcal{W}_{pq} = \{0\}$), $\forall\, p, q \in \mathcal{P}$,
2) $\min_{t\geq t_0} |W_p \widehat{Y}(t)| \geq \beta > \varrho$ for each $p \in \mathcal{P}$.

If $\delta_0 < \frac{\Omega}{e^{(a+ka)}}$, and $x(t_0) \in \mathcal{B}_{\delta_0}(\hat{x}(t_0))$, then $(\hat{x}(t), \hat{y}(t))$ do not form a $(\mathcal{R}_t, \varrho)$ switch-singular pair for any $t \in [t_0, T)$, where

$$T < t_0 + \frac{1}{\lambda}\log\left(\frac{\lambda\Omega + e^{(a+ka)}b\varrho}{\lambda\delta_0 + b\varrho}\right) - \frac{(k+1)a}{\lambda}, \tag{13}$$

and k is the number of switches over the interval $[t_0, T)$. $\lhd$

Based on the result of Theorem 10, one can formulate an alternative *index-matching function* $\widetilde{\Sigma}^{-1}$ as follows:

$$\widetilde{\Sigma}^{-1}(\hat{x}(t), y_{[t,t+\epsilon)}) = \{p : |L_p\hat{x}(t) - W_p\widehat{Y}(t)| \leq \|L_p\|\delta(t) + \|W_p\|\varrho\}. \tag{14}$$

If we compare the two functions $\widehat{\Sigma}^{-1}$ in (8) and $\widetilde{\Sigma}^{-1}$ in (14), then it is observed that the mode detection through $\widehat{\Sigma}^{-1}$ requires the computation of minimum distance between the reachable sets and the measured output at each instant in time whereas the function $\widetilde{\Sigma}^{-1}$ only requires coarse spherical approximation of the reachable set which can be obtained analytically. The interval over which the switching signal can be constructed is, in general, larger with $\widehat{\Sigma}^{-1}$ than with $\widetilde{\Sigma}^{-1}$. To obtain larger time interval for reconstruction of switching signal with light computation, one may combine the index-matching function $\widetilde{\Sigma}^{-1}$ in (14) with the computation of tightly approximated reachable sets. This can be done by resetting the value of $\delta(t)$, at regular intervals, to a number that tightly approximates the radius of reachable sets.

C. Input Recovery

The input reconstructed using the measured output is given by (6b). Using the exact expression for u in (5c), the input estimation error, $\tilde{u} := u - \hat{u}$, is given by

$$\tilde{u}(t) = (-\overline{D}_\alpha^{-1}\overline{C}_\alpha)_{\sigma(t)}\tilde{x}(t) + (\overline{D}_\alpha^{-1}V)_{\sigma(t)}(\widehat{Y}(t) - Y(t)).$$

Using the notation, $d_c := \max_{p\in\mathcal{P}} \|(\overline{D}_\alpha^{-1}\overline{C}_\alpha)_p\|$ and $d_v := \max_{p\in\mathcal{P}} \|(\overline{D}_\alpha^{-1}V)_p\|$, the maximal error in the reconstruction of u at any time t is given by

$$|\tilde{u}(t)| \leq d_c\delta(t) + d_v\varrho. \tag{15}$$

IV. Minimum Phase Systems

In the previous section, the results were stated for a general class of linear systems without any stability assumptions. In classical linear systems theory, the stability of the inverse system is closely related to the minimum phase property of the system. This idea is now employed to derive conditions for recovering the switching signal over the interval $[t_0, \infty)$.

For each subsystem Γ_p, the matrix L_p has r_p rows and rank r_p. So, there exists an $(n - r_p) \times n$ matrix $\underline{T}_p$ such that $T_p := \left[\begin{smallmatrix} L_p \\ \underline{T}_p \end{smallmatrix}\right]$ and $L_p T_p^{-1} = [I_{r_p\times r_p}\ 0_{r_p\times(n-r_p)}]$. The matrix T_p defines a coordinate transformation for the subsystem Γ_p and the transformed matrices are: $A_p^* := T_p A_p T_p^{-1}$, $B_p^* := T_p B_p$, $C_p^* := C_p T_p^{-1}$, $D_p^* := D_p$, and $L_p^* = [I_{r_p\times r_p}\ 0_{r_p\times(n-r_p)}]$. Apply the structure algorithm in the new coordinates, and let F_p be the matrix formed from the last $n - r_p$ rows and columns of $(A_p^* - B_p^*\overline{D}_{\alpha_p}^{*-1}\overline{C}_{\alpha_p}^*)$, let G_p^1 be the matrix formed from the first r_p columns and the last $(n - r_p)$ rows of $(A_p^* - B_p^*\overline{D}_{\alpha_p}^{*-1}\overline{C}_{\alpha_p}^*)$, and let G_p^2 be the matrix formed by the last $(n - r_p)$ rows of $B_p^*\overline{D}_{\alpha_p}^{*-1}$. If $z_p := T_p x$ denotes the new state variable, then

$$z_p^1 := [(z_p)_1, \cdots, (z_p)_{r_p}]^\top = [\tilde{y}_0^\top, \cdots, \tilde{y}_{\alpha_p-1}^\top]$$

and for the remaining $(n - r_p)$ state variables denoted by z_p^2, the dynamical equation is:

$$\dot{z}_p^2 = F_p z_p^2 + G_p^1 z_p^1 + G_p^2 \overline{y}_{\alpha_p}.$$

Let $\hat{z}_p$ be an estimate of z_p, and let $\tilde{z}_p(t) := \hat{z}_p(t) - z_p(t)$ denote the error between the actual state trajectory and the simulated one, then $|\tilde{z}_p^1| \leq \varrho$ and using $\overline{y}_{\alpha_p} = V_p Y$,

$$\dot{\tilde{z}}_p^2 = F_p\tilde{z}_p^2 + G_p w,$$

where $G_p = [G_p^1\ G_p^2 V_p]$ and $|w| \leq \varrho$.

Definition 11 (Minimum phase system): The subsystem Γ_p is called minimum-phase if F_p is Hurwitz. $\lhd$

Under minimum-phase assumption on each subsystem, we first show that there is a uniform bound on maximal uncertainty in the reachable sets at all times under the dwell-time assumption. To see this, note that if Γ_p is minimum-phase, then there exists an $(n - r_p) \times (n - r_p)$ matrix P_p such that $V_p : \mathbb{R}^{r_p} \to \mathbb{R}$ defined as $V_p(\tilde{z}_p^2) = \tilde{z}_p^{2\top} P_p \tilde{z}_p^2$ is a Lyapunov function for $\tilde{z}_p^2$ and there exists a positive definite matrix Q_p such that $F_p^\top P_p + P_p F_p = -Q_p$. With $\Theta_p := \frac{2\|P_p G_p\|}{\lambda_{\min}(Q_p)}$, and $\varepsilon > 0$ small enough, the inequality $|\tilde{z}_p^2| > \Theta_p\varrho(1 + \varepsilon)$

implies $\dot{V}_p < -|\tilde{z}_p^2|\lambda_{\min}(Q_p)\Theta_p\varrho\varepsilon$. Introducing some more notation, define $\overline{P}_p := \left[\begin{array}{c|c} I_{r_p} & 0 \\ \hline 0 & P_p \end{array}\right]$ for each $p \in \mathcal{P}$; also let $\Theta := \min_{p\in\mathcal{P}} \Theta_p$; $\hat{\delta} := \min\{\varrho, \Theta\varrho(1+\varepsilon)\}$, and $\delta := \frac{\hat{\delta}}{\|T\|}$, where $\|T\| := \max_{p\in\mathcal{P}} \|T_p\|$. Let τ_d be defined as:

$$\max_{p,q\in\mathcal{P}} \frac{(\overline{\lambda}(M_{qp})-1)\varrho^2 + \varrho^2(1+\varepsilon)^2(\overline{\lambda}(M_{qp})\overline{\lambda}(P_p)\Theta_p^2 - \overline{\lambda}(P_q)\Theta_q^2)}{\Theta_q^2\varrho^2(1+\varepsilon)\lambda_{\min}(Q_q)\varepsilon}, \tag{16}$$

where $M_{qp} := H_{qp}^{-1^\top}\overline{P}_q H_{qp}^{-1}$, H_{qp} is an upper triangular matrix satisfying $T_{qp}^\top \overline{P}_p T_{qp} = H_{qp}^\top H_{qp}$, $T_{qp} := T_p T_q^{-1}$, and $\overline{\lambda}(M)$ denotes the maximum eigenvalue of a matrix M. The uniform bound for the state trajectories, under the slow-switching assumption with dwell-time τ_d, comes out to be:

$$\Delta := \max_{p,q\in\mathcal{P}} \frac{\lambda_{\max}(M_{qp})}{\lambda_{\min}(\overline{P}_q)}(\varrho^2 + \lambda_{\max}(P_p)\Theta_p^2\varrho^2(1+\varepsilon)^2). \tag{17}$$

Proposition 12: Consider system (1) and assume that Γ_p is minimum phase for each $p \in \mathcal{P}$, and $\varrho > 0$ is such that $|Y(t) - \widehat{Y}(t)| \le \varrho$ for each t. Then $x(t) \in \mathcal{B}_\Delta(\hat{x}(t))$ for all $t \ge t_0$, provided the initial state $x(t_0) \in \mathcal{B}_\delta(\hat{x}_0)$ and $t_{i+1} - t_i \ge \tau_d$, for every switching instant t_i. $\lhd$

The result conceptually relates to the incremental input-to-state stability property of the system (10), which has been studied in [12] for homogenous systems. But here, the formulation takes into account the disturbances due to measurement uncertainties and the bounds on input-to-state gains are also computed. The proof uses the level sets of Lyapunov functions to derive bounds on system trajectories, similar to [11, Chapter 5]; and in the process, following lemma is employed to make the bounds tighter.

Lemma 13: Given two positive definite functions $V_1 = x^\top P_1 x$, and $V_2 = x^\top P_2 x$ with P_1 and P_2 symmetric positive definite matrices, the minimal level set of V_2 that contains the set $\{x \,|\, V_1(x) \le c\}$ is given by $\{x \,|\, V_2(x) \le \lambda_{\max}(H^{-1^\top}P_2H^{-1})c\}$, where the matrix H is an upper triangular matrix that satisfies $P_1 = H^\top H$.

Proof: The matrix P_1 admits Cholesky decomposition given by $P_1 = H^\top H$, where H is an upper triangular matrix. It follows that $H^{-1^\top}P_1H^{-1} = I$. Let $z = Hx$; in the new coordinates defined by z, the level sets of V_1 are spheres of dimension $n-1$, with radius c. Consider the region $\mathcal{R} := \{z \,|\, z^\top H^{-1^\top}P_2H^{-1}z \le \lambda_{\max}(H^{-1^\top}P_2H^{-1})c\}$. If $|z|^2 \le c$, then $z \in \mathcal{R}$. Moreover, if z is in the span of eigenvector corresponding to $\lambda_{\max}(H^{-1^\top}P_2H^{-1})$ with $|z|^2 = c$, then z is also on the boundary of $\mathcal{R}$ implying that the bounding region $\mathcal{R}$ wraps the level sets of V_1 tightly. Applying the transformation, $x = H^{-1}z$ gives the desired result. ■

Using the bound in Proposition 12, a result parallel to Theorem 10 is now stated.

Theorem 14: For system (1), if each subsystem Γ_p is minimum phase and moreover,

1) condition (3) holds (i.e., $\mathcal{L}_{pq} \cap \mathcal{W}_{pq} = \{0\}$), $\forall\, p,q \in \mathcal{P}$,
2) the measured output $\hat{y}$ is such that $\min_{t\ge t_0} |W_p\widehat{Y}(t)| \ge \beta > \varrho$ for each $p \in \mathcal{P}$,
3) the inequality $\Delta < \Omega$ holds, where Δ is given in (17), and Ω in (12),
4) the dwell-time of σ is given by τ_d in (16).

Then $\sigma(t) = \{p : |L_p\hat{x}(t) - W_p\widehat{Y}(t)| \le \|L_p\|\Delta + \|W_p\|\varrho\}$ for all $t \ge t_0$. Moreover, $\|\tilde{u}\|_\infty = d_c\Delta + d_v\varrho$. $\lhd$

The proof uses the bound in Proposition 12 that $x(t) \in \mathcal{B}_\Delta(\hat{x}(t))$. Condition 3 of the theorem statement implies that the inequality (11) is violated for all times and this in turn implies that $(\hat{x}(t), \hat{y}(t))$ does not form a $(\mathcal{B}_\Delta(\hat{x}(t)), \varrho)$ switch-singular pair for any $t \ge t_0$. Thus, the index-matching function of (14) is well-defined and it reconstructs the original switching signal. The uniform upper bound on $\tilde{u}$ is obtained from (15).

Appendix

In the construction of inverse systems via the structure algorithm given in [1], there are two differential operators acting on the output y, which are denoted by $\overline{N}_\alpha$ and $\mathbf{N}$. Below we seek a simpler representation so that $\mathbf{N}y$ and $\overline{N}_\alpha y$ can be written as a matrix (with real coefficients) times a vector (comprising of output and its derivatives), i.e., $\mathbf{N}y = WY$ and $\overline{N}_\alpha y = VY$, for some matrices W and V. Following the notation of [1, Page 952-953], let $\widetilde{G}_i := \mathrm{col}\left(\widetilde{S}_0K_{-1,i}, \widetilde{S}_1K_{0,i}, \cdots, \widetilde{S}_{\beta-1}K_{\beta-2,i}\right)$, $\overline{G}_j := \overline{S}_\alpha K_{\alpha-1,j}$, $0 \le i \le \beta-1$, $0 \le j \le \alpha$, $K_{-1,0} = I$, and $K_{j,k} = 0\ \forall k \ge j+2$, $\forall\, j$. We can then write

$$\mathbf{N}y =: \widetilde{G}_0 y + \frac{d}{dt}\big(\widetilde{G}_1 y + \cdots + \frac{d}{dt}\big(\widetilde{G}_{\beta-2}y + \frac{d}{dt}\widetilde{G}_{\beta-1}y\big)\big),$$
$$\overline{N}_\alpha y =: \overline{G}_0 y + \frac{d}{dt}\big(\overline{G}_1 y + \cdots + \frac{d}{dt}\big(\overline{G}_{\alpha-1}y + \frac{d}{dt}\overline{G}_\alpha y\big)\big).$$

We now define the desired matrices W and V as, $W := [\ \widetilde{G}_0 \ \cdots\ \widetilde{G}_{\beta-1}\ \ \widetilde{G}_\beta\ \cdots\ \widetilde{G}_n\ ]$ and $V := [\ \overline{G}_0\ \cdots\ \overline{G}_\alpha\ \ \overline{G}_{\alpha+1}\ \cdots\ \overline{G}_n\ ]$, where $\widetilde{G}_i = 0$ for $\beta \le i \le n$ and $\overline{G}_i = 0$ for $\alpha+1 \le i \le n$.

References

[1] L. Vu and D. Liberzon, "Invertibility of switched linear systems," *Automatica*, vol. 44, no. 4, pp. 949–958, April 2008.
[2] A. Tanwani and D. Liberzon, "Invertibility of switched nonlinear systems," *Automatica*, vol. 46, no. 12, pp. 1962 – 1973, 2010.
[3] M. Fliess, C. Join and H. Sira-Ramírez , "Non-linear estimation is easy," *Intl. Jnl. of Modeling, Ident. & Control*, vol. 14, no. 1, pp. 12–27, 2008.
[4] L. M. Silverman, "Inversion of multivariable linear systems," *IEEE Trans. Automat. Contr.*, vol. 14, no. 3, pp. 270–276, 1969.
[5] L. M. Silverman and H. J. Payne, "Input-output structure of linear systems with aplication to decoupling problem," *SIAM Jnl. on Contr. & Optimiz.*, vol. 9, no. 2, pp. 199–233, 1971.
[6] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*. Kluwer, Dordrecht, 1988.
[7] A. Kurzhanski and P. Varaiya, "Ellipsoidal techniques for reachability analysis. Parts I & II," *Optimiz. Meth. & Soft.*, vol. 17, pp. 177–206 and 207–237, 2000.
[8] D. G. Luenberger, *Optimization by Vector Space Methods*. John Wiley & Sons, Inc., New York, 1969.
[9] J. Gross and G. Trenkler, "On the least squares distance between affine subspaces," *Lin. Alg. & Appl.*, vol. 237-238, pp. 269 – 276, 1996.
[10] C.-Y. Deng, "On the minimum gap and the angle between two subspaces," *J. Comput. Appl. Math.*, vol. 206, pp. 625–630, 2007.
[11] D. Liberzon, *Switching in Systems and Control*. Birkhäuser, 2003.
[12] A. Girard, G. Pola and P. Tabuada, "Approximately bisimilar symbolic models for incrementally stable switched systems," *IEEE Trans. Automat. Contr.*, vol. 55, no.1, pp.116–126, 2010.